Measure Theory with Ergodic Horizons
Lecture 30

We would now like to characterize those functions $f : \mathbb{R} \to \mathbb{R}$ which are distributions of loc. finite Borel measures μ on $\mathbb{R}$ with $\mu \ll \lambda$. We already know that such a function is right-continuous and increasing, and $\mu \ll \lambda$, together with local finiteness of μ, gives: $\forall (a,b)$ $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall$ $B \subseteq (a,b)$

$$\lambda(B) \leq \delta \implies \mu(B) \leq \varepsilon.$$

For open B, we know that $B = \bigsqcup_{n \in \mathbb{N}} (a_n, b_n)$ so $\lambda(B) = \sum_{n \in \mathbb{N}} (b_n - a_n)$ and $\mu(B) = \sum_{n \in \mathbb{N}} \mu((a_n, b_n))$ but μ has no atoms by $\mu \ll \lambda$, so $\mu((a_n, b_n)) = \mu((a_n, b_n]) = f(b_n) - f(a_n)$, so $\mu(B) = \sum_{n \in \mathbb{N}} (f(b_n) - f(a_n)) = \sum_{n \in \mathbb{N}} |f(b_n) - f(a_n)|$. Thus, we have

$$\lambda(B) = \sum_{n \in \mathbb{N}} (b_n - a_n) \leq \delta \implies \mu(B) = \sum_{n \in \mathbb{N}} |f(b_n) - f(a_n)| \leq \varepsilon.$$

This motivates the following:

Def. A function $f : \mathbb{R} \to \mathbb{R}$ is called *absolutely continuous* on (a,b) if $\forall \varepsilon > 0$ $\exists \delta > 0$ for all open $\bigsqcup_{n \in \mathbb{N}} (a_n, b_n) \subseteq (a,b)$, we have

$$\sum_{n \in \mathbb{N}} (b_n - a_n) \leq \delta \implies \sum_{n \in \mathbb{N}} |f(b_n) - f(a_n)| \leq \varepsilon.$$

We say that f is *locally absolutely continuous* if f is absolutely continuous on every $(a,b) \subseteq \mathbb{R}$.

Examples. Lipschitz $\Rightarrow$ absolutely continuous $\Rightarrow$ uniformly continuous.

Theorem (characterization of distributions of $\mu \ll \lambda$). For an increasing $f: \mathbb{R} \to \mathbb{R}$, TFAE:

(1) f is a distribution of a loc. finite Borel measure $\mu \ll \lambda$,

(2) FTC holds for f, i.e. f' exists a.e. and $f(b) - f(a) = \int_a^b f' \, d\lambda$ for all $a < b$.

(3) f is locally absolutely continuous.

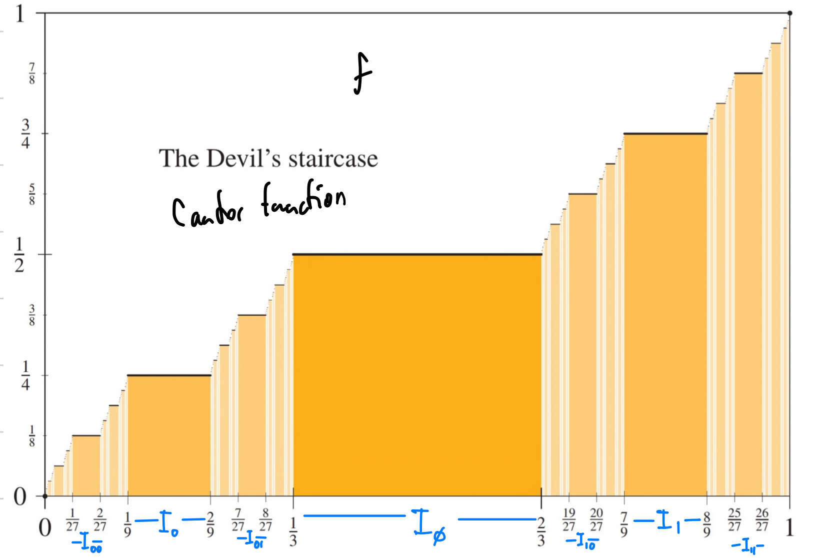
Proof. (1) $\Leftrightarrow$ (2) was proved last time and (1) $\Rightarrow$ (3) was argued above by restricting $\mu \ll \lambda$ to open sets. We show (3) $\Rightarrow$ (1) by the regularity of μ. Suppose (3).
Since f is increasing and continuous, it is a distribution of a loc. finite nonatomic loc. finite Borel measure μ on $\mathbb{R}$. To show that $\mu \ll \lambda$, fix a λ-null set $B \subseteq (a,b)$ and show that $\mu(B) = 0$, or equivalently, $\mu(B) \leq \varepsilon$ for all $\varepsilon > 0$.
Let $\delta > 0$ be given by absolute continuity on (a,b) for this ε. By regularity of λ, there is an open set $U \supseteq B$ such that $\lambda(U) \leq \delta$. But then $U = \bigsqcup_{n \in \mathbb{N}} (a_n, b_n)$,
so $\lambda(U) = \sum_{n \in \mathbb{N}} (b_n - a_n) \leq \delta$ hence $\mu(B) = \sum_{n \in \mathbb{N}} \mu((a_n, b_n)) = \sum_{n \in \mathbb{N}} \mu([a_n, b_n]) = \sum_{n \in \mathbb{N}} f(b_n) - f(a_n) \leq \varepsilon$.

μ is nonatomic

$\square$

We now give an example of a uniformly continuous increasing function that is not absolutely continuous.

Example (the devil's staircase). Let $\varphi : 2^{\mathbb{N}} \to \mathbb{R}$ by mapping binary sequences to ternary representations: $x \mapsto 0.(2\cdot x_0)(2\cdot x_1)(2\cdot x_2)\ldots$ Thus, φ is a homeomorphism between $2^{\mathbb{N}}$ and the standard Cantor set C. Let μ on $\mathbb{R}$ be the pushforward measure of Bernoulli$(\frac{1}{2})$ by φ. Then the distribution $f(x) := \mu((0,x])$ of μ is a continuous increasing function $f : [0,1] \to [0,1]$, in particular, uniformly continuous because $[0,1]$ is compact. This f is called the Cantor function or devil's staircase.



We know that $U := [0,1] \setminus C$ is μ-null hence f is constant on every interval $I \subseteq U$. Here is the explicit description of f. Write $x \in [0,1]$ in ternary, favouring 1's, i.e. write $0.2010000\ldots$ instead of $0.2002222\ldots$ Then cut all the sequence to the right of the first 1, if exists, and turn all 2's into 1's. So

$$f(0.2012202\ldots) = 0.101 \quad \text{and} \quad f(0.2020220\ldots) = 0.1010110\ldots$$

Note that indeed, since $\mu \perp \lambda$, $f' = 0$ a.e. (which we already knew from the fact that f is constant on every interval $\subseteq U$) and f is not absolutely continuous by (3)$\Rightarrow$(1) of the theorem above.

Lastly, we would like to characterise (maybe nonincreasing) functions for which the FTC holds, and for this we have to consider signed measures on $\mathbb{R}$. Because non-finite signed measures are slightly annoying to handle, we restrict to finite signed measures.

Def. For a signed measure ν on some measurable space, let $\nu = \nu_+ - \nu_-$ be its Hahn decomposition into a difference of measures. Denote $\nu_* := \nu_+ + \nu_-$ and call it the *total variation* of ν. We say that ν is finite if ν_* is a finite measure.

We already know that

finite Borel measures $\xleftrightarrow{\text{distribution}}$ bounded increasing right-continuous functions

finite Borel signed measures $\longleftrightarrow$?

For a finite signed Borel measure ν on $\mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ is called a *distribution* of ν if $\nu((a,b]) = f(b) - f(a)$. For example, $f(x) := \begin{cases} \nu((0,x]) & \text{if } x > 0 \\ \nu((x,0]) & \text{if } x \leq 0 \end{cases}$.

These functions are unique up to a constant.

We know that μ finite measure has bdd distribution. We need to come up with the correct analogue of "bdd" for signed measures. Let ν be a finite signed measure on $\mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a distribution of ν. By Hahn decomposition of $\nu = \nu_+ - \nu_-$, we know that $f = f_+ - f_-$, where f_+, f_- are the respective distributions of ν_+, ν_-. So f is a difference of two increasing functions, each of which is bdd. We would like to understand how to find this f_+ and f_-.

By the finiteness of ν, $\nu^*(B) \leq \nu^*(\mathbb{R}) < \infty$. Thus, for any open $U = \bigsqcup_{n\in\mathbb{N}} (a_n, b_n)$,

$$\infty > \nu^*(\mathbb{R}) \geq \nu^*(U) \geq \sum_n \nu^*((a_n, b_n]) \geq \sum_n |\nu((a_n, b_n])| = \sum_n |f(b_n) - f(a_n)|.$$

This motivates the following:

Def. For $f: \mathbb{R} \to \mathbb{R}$, let $T_f: \mathbb{R} \to [0, \infty]$ be defined by:

$$T_f(x) := \sup \left\{ \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)| : -\infty < x_0 < x_1 < \dots < x_n \leq x \right\}.$$

T_f is called the total variation of f. We say that f has bounded variation if $T_f(\infty) := \lim_{x\to\infty} T_f(x) < \infty$.

Note. For $a < b$, $T_f(b) - T_f(a) = \sup \left\{ \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)| : a \leq x_0 < x_1 < \dots < x_n \leq b \right\}$ is the vertical distance traveled by the graph of f on (a, b).

Prop. For $f: \mathbb{R} \to \mathbb{R}$, TFAE:

(1) f has bdd total variation.

(2) f is a difference of bdd increasing functions, namely $f = \frac{1}{2}(T_f + f) - \frac{1}{2}(T_f - f)$.

Proof. HW.

We can now fill in the ? in "finite Borel signed measures $\longleftrightarrow$?"

Theorem. For $f:\mathbb{R}\to\mathbb{R}$, TFAE:

(1) f is a distribution of a finite signed Borel measure ν on $\mathbb{R}$.

(2) f is right-continuous and has bdd total variation.

Proof. We have already shown (1)$\Rightarrow$(2) by Hahn decomposition of ν, and (2)$\Rightarrow$(1) is HW. $\square$

We can also prove the analogue of FTC for (potentially nonincreasing) functions:

Theorem. For a function $f:\mathbb{R}\to\mathbb{R}$, TFAE:

(1) f is a distribution of a finite signed Borel measure ν with $\nu^* \ll \lambda$.

(2) f' exists a.e. and is in $L^1(\mathbb{R},\lambda)$, and the FTC holds: $f(b)-f(a)=\int_a^b f'\,d\lambda \;\forall a<b$.

(3) f had bdd total variation and is absolutely continuous.

Proof-sketch. This follows from the analogous theorem for increasing functions and the fact that bdd total variation is equivalent to being a difference of two bdd increasing functions. Details left as HW. $\square$